Int J Theor Phys (2008) 47: 2133–2140
DOI 10.1007/s10773-007-9571-3

# Estimation Strategies for Finite Dimensional Systems

**Chiara Macchiavello**

Received: 6 December 2006 / Accepted: 14 September 2007 / Published online: 9 October 2007


**Abstract** We review the performance of state estimation procedures for quantum states with arbitrary finite dimension. We compare the cases of universal and multi-phase covariant estimation. We discuss the form of the optimal measurements in various cases when a single system is available.

**Keywords** Quantum measurements · Quantum information

## 1 Introduction

The limits to the precision achievable in the estimation of unknown quantum states represent a topic of fundamental interest.[1] Moreover, the detection process is usually the final step in any quantum information task and therefore the issue of measuring quantum states in the most efficient way is a central one. The efficiency of the estimation process depends crucially on the a priori information available about the states to be estimated. In this work we review some results related to the limits on estimation procedures for finite dimensional systems in the cases where no a priori information is available (universal case) and where some information about the form of the states is known and only some phases need to be estimated (phase covariant case). We stress that the latter case in particular has relevant applications in quantum computation and quantum information. For example, it was shown that the existing quantum algorithms can be described in a unified way as quantum interference processes among different computational paths where the result of the computation is encoded in a phase shift [2]. Moreover, the design of optimal phase measurement procedures is also crucial in various tasks of atomic physics, such as for example methods for precision spectroscopy [3], and quantum interferometric experiments in quantum optics.

C. Macchiavello (✉)
Dipartimento di Fisica "A.Volta" and IUSS, Via Bassi 6, 27100 Pavia, Italy
e-mail: chiara@unipv.it

[1] For a pedagogical review on the main concepts in state discrimination and state estimation see [1].

We also want to point out that the possibility of employing quantum systems with (finite) dimension higher than two in quantum information been recently triggered much interest. In particular, it has been shown that an increase in the dimension leads to a better performance of various quantum information protocols, such as for example quantum cryptography [4–6] and some problems in distributed quantum computing [7]. Moreover, a relevant experimental effort has been recently spent in the generation, manipulation and detection of quantum systems with higher dimension [8, 9].

The paper is organised as follows. In Sect. 2 we review the efficiency limits for state estimation in the universal case. In Sect. 3 we consider the multiple phase covariant case and present a discretised solution for the optimal measurement strategy. In Sect. 4 we compare the explicit construction of the measurement procedures in the two cases for a single system in finite dimension $d$ (qudit), and analyse for completeness also the case of single phase estimation on a qudit system. Finally, we close the paper by summarising the main results in Sect. 5.

## 2 Universal Estimation of Qudits

In this section we consider the case where we are given $N$ copies of a completely unknown pure quantum state $|\psi\rangle$ with arbitrary finite dimension $d$ and we briefly review recent results about the limits of its estimation as functions of $N$ and $d$. We are interested here in universal estimation, namely we optimise the efficiency of the estimation procedure by averaging over all possible input states. We study the efficiency in terms of the fidelity of the reconstructed state, which is given by

$$F_u(N) = \int d\Omega_\psi \sum_\mu p_\mu(\psi)|\langle\psi|\bar{\psi}_\mu\rangle|^2, \tag{1}$$

where $p_\mu(\psi)$ is the probability of finding outcome $\mu$ (to which we associate candidate $|\bar{\psi}_\mu\rangle$ as the best guess in the estimation), given that the inputs were in state $|\psi\rangle$, and the integration is performed to average over all possible input states. The above fidelity quantifies how close, on average, the guessed state is to the actual one. Following the original approach of [18], we notice that this problem is related to the problem of universal cloning for finite dimensional systems [16], where $N$ copies of an unknown input states are transformed into $M$ output copies. If we denote by $F_{\mathrm{cl}}^{\mathrm{opt}}(N, M)$ the optimal cloning fidelity of the input–output single copy, the following relation with the optimal fidelity for state estimation holds [18]

$$F_{\mathrm{cl}}^{\mathrm{opt}}(N,\infty) = F_{\mathrm{u}}^{\mathrm{opt}}(N), \tag{2}$$

namely the optimal state estimation fidelity corresponds to the optimal cloning fidelity in the limit of an infinite number of output copies. The relation (2) was later proved to hold for the multi-phase covariant case [10], which will be addressed in the following section, and recently its general validity was shown [11]. Therefore, it provides a useful tool to establish the efficiency limits in the performance of state estimation from cloning properties, or vice versa.

In the universal case, by exploiting the results of [16] on optimal cloning transformations and the above relation, the optimal fidelity for universal state estimation turns out to be of the form

$$F_{\mathrm{u}}^{\mathrm{opt}}(N) = \frac{N+1}{N+d}. \tag{3}$$

In the next section we will compare the above expression with the optimal fidelity for multiple phase estimation.

## 3 Multiple Phase Estimation

In this section we assume that some a priori information about the states to be estimated is available and assume in particular that they are "equatorial states", defined as

$$|\psi(\{\phi_j\})\rangle = \frac{1}{\sqrt{d}}(|0\rangle + e^{i\phi_1}|1\rangle + e^{i\phi_2}|2\rangle + \cdots + e^{i\phi_{d-1}}|d-1\rangle), \tag{4}$$

where $\{|0\rangle, |1\rangle, |2\rangle, \ldots, |d-1\rangle\}$ represents a basis for the system under consideration and $\{\phi_j\}$ denotes a set of $d-1$ independent phase-shifts ($\phi_j \in [0, 2\pi]$). The estimation problem then reduces to optimally estimate $d-1$ independent phase shifts and can then be cast in the framework of quantum estimation theory for multiple phase shifts induced by commuting operators on a quantum state [13]. The formulation of the multiple phase estimation problem [13] is a generalisation of the single phase estimation problem addressed in [12, 15] and extended in [14]. We briefly review the main steps here in the specific case of $N$ identically prepared equatorial states (4). We will find the optimal form for the reconstruction fidelity and also the explicit form of the optimal measurement procedure in terms of positive-operator valued measurement (POVM) [12].

The efficiency of the multi-phase estimation procedure is evaluated in general in terms of a cost function $C(\{\bar{\phi}_j\}, \{\phi_j\})$ which depends on the set of the $d-1$ estimated values $\{\bar{\phi}_j\}$, that are the results of the estimation procedure, and on the set of the $d-1$ true values $\{\phi_j\}$. The cost function quantifies the errors for the estimates $\{\bar{\phi}_j\}$ given the true values $\{\phi_j\}$. By assuming that the values of the phase shifts are completely unknown and they are a priori uniformly distributed in the interval $[0, 2\pi]$, the estimation problem then consists in minimizing the average cost $\bar{C}$ of the procedure, defined in this case as

$$\bar{C} = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \cdots \int_0^{2\pi} \frac{d\phi_{d-1}}{2\pi} \int_0^{2\pi} d\bar{\phi}_1 \cdots \int_0^{2\pi} d\bar{\phi}_{d-1} C(\{\bar{\phi}_j\}, \{\phi_j\}) p(\{\bar{\phi}_j\}|\{\phi_j\}). \tag{5}$$

In the above expression $p(\{\bar{\phi}_j\}|\{\phi_j\})$ is the conditional probability of estimating the set of values $\{\bar{\phi}_j\}$ given the true values $\{\phi_j\}$, namely $p(\{\bar{\phi}_j\}|\{\phi_j\}) d\bar{\phi}_1 \cdots d\bar{\phi}_{d-1} = \mathrm{Tr}[d\mu(\{\bar{\phi}_j\})|\psi(\{\phi_j\})\rangle\langle\psi(\{\phi_j\})|]$ and $d\mu(\{\bar{\phi}_j\})$ denotes the POVM.

In this work we consider the case where the errors in the estimates are weighted independently of the values $\phi_j$ of the phases, but they depend only on the values of the differences $\bar{\phi}_j - \phi_j$, so that the cost function becomes an even function of $d-1$ variables, i.e. $C(\{\bar{\phi}_j\}, \{\phi_j\}) \equiv C(\{\bar{\phi}_j - \phi_j\})$. From these requirements it follows that also the conditional probability corresponding to the optimal estimation procedure will depend only on the variables $\bar{\phi}_j - \phi_j$, and therefore the optimal POVM can be considered of the phase-covariant form

$$d\mu(\{\bar{\phi}_j\}) = e^{-i\sum_{j=1}^{d-1} \bar{\phi}_j \hat{H}_j} \chi e^{i\sum_{j=1}^{d-1} \bar{\phi}_j \hat{H}_j} \frac{d\bar{\phi}_1}{2\pi} \frac{d\bar{\phi}_2}{2\pi} \cdots \frac{d\bar{\phi}_{d-1}}{2\pi}, \tag{6}$$

where $\hat{H}_j = \sum_{k=1}^{N} |j\rangle\langle j|_k$, and $|j\rangle\langle j|_k$ denotes the projection operator onto the state $|j\rangle$ of the $k$th qudit. In the above equation $\chi$ is a positive operator satisfying the completeness constraints needed for the normalization of the POVM $\int d\mu(\{\phi_j\}) = \mathbb{1}$, where $\mathbb{1}$ denotes the identity operator.

The cost functions usually considered are $2\pi$-periodic functions in the variables $\{\phi_j\}$, and therefore they can be written as

$$C(\{\phi_j\}) = -\sum_{l_1,l_2,\ldots,l_M=-\infty}^{\infty} c_{\{l_j\}} e^{i\sum_j l_j\phi_j}, \tag{7}$$

with the condition $c_{\{l_j\}} = c_{\{-l_j\}}$ due to the fact that the cost is a real and even function. We will now focus on a general class of cost functions, that extends the one considered by Holevo [15], with

$$c_{\{l_j\}} \geq 0, \quad \forall\{l_j\} \neq 0. \tag{8}$$

For the present case of $N$ identical equatorial qudits the input state belongs to the symmetric subspace and therefore we can restrict our description of the POVM to the symmetric basis $\{|n_0, n_1, n_2, \ldots, n_{d-1}\rangle_s, \sum_{j=0}^{d-1} n_j = N\}$, where $|n_0, n_1, n_2, \ldots, n_{d-1}\rangle_s$ is the symmetric state of $N$ qudits, with $n_0$ qudits in state $|0\rangle$, $n_1$ in state $|1\rangle$, and so on (for more details on the rigorous derivation of the optimal POVM see [13]).

For the Holevo class the average cost is given by

$$\begin{aligned}\bar{C} = -c_0 - \sum_{\{l_j\}} c_{\{l_j\}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \cdots \int_0^{2\pi} \frac{d\phi_{d-1}}{2\pi} e^{i\sum_j l_j\phi_j} \\ \times \sum_{\{n_j\},\{m_j\}} e^{i\sum_j (m_j-n_j)\phi_j} {}_s\langle\{n_j\}|\Psi_0\rangle\langle\Psi_0|\{n_j\}\rangle_s \chi_{\{n_j\},\{m_j\}},\end{aligned} \tag{9}$$

where $\chi_{\{n_j\},\{m_j\}}$ denote the POVM matrix elements in the symmetric basis defined above, and $|\Psi_0\rangle$ is the state to be estimated assuming that all the phase shifts vanish. In the present scenario of $N$ identical equatorial states

$$|\Psi_0\rangle = |\psi(\{\phi_j = 0\})\rangle^{\otimes N} = \frac{1}{\sqrt{d^N}} \sum_{\{n_j\}} \sqrt{\frac{N!}{n_0!n_1!n_2!\cdots n_{d-1}!}} |n_0, n_1, n_2, \ldots, n_{d-1}\rangle_s, \tag{10}$$

but more generally $|\Psi_0\rangle$ can be any symmetric state of $N$ qudits. By performing the integrals in (9), exploiting the relation $\int_0^{2\pi} d\phi e^{i(n-m)\phi} = 2\pi\delta_{n,m}$ and by performing the minimisation in (9) on the POVM elements $\chi_{\{n_j\},\{m_j\}}$, we arrive at the following expression for the minimum cost

$$\bar{C} = -c_0 - \sum_{\{l_j\}\neq 0} c_{\{l_j\}} \sum_{\{m_j-n_j\}=\{l_j\}} |\langle\Psi_0|\{n_j\}\rangle_s||_s\langle\{m_j\}|\Psi_0\rangle|. \tag{11}$$

This minimum value is achieved by the optimal POVM

$$d\mu(\{\phi_j\}) = \frac{d\phi_1}{2\pi} \cdots \frac{d\phi_{d-1}}{2\pi} |e(\{\phi_j\})\rangle\langle e(\{\phi_j\})|, \tag{12}$$

where the vectors $|e(\{\phi_j\})\rangle$ are defined as

$$|e(\{\phi_j\})\rangle = \sum_{\{n_j\}} e^{i\sum_{j=1}^{d-1} n_j\phi_j} |n_0, n_1, n_2, \ldots, n_{d-1}\rangle_s. \tag{13}$$

We want to point out that the minimum cost (11) can be achieved also by means of a "discretized" POVM

$$\hat{P} = \frac{1}{(N+1)^{d-1}} \sum_{\{k_j\}} |e(\{\phi_j(k_j)\})\rangle\langle e(\{\phi_j(k_j)\})|, \tag{14}$$

namely a finite set of projectors defined by the vectors

$$|e(\{\phi_j(k_j)\})\rangle = \sum_{\{n_j\}} e^{i\sum_{j=1}^{d-1} n_j\phi_j(k_j)} |n_0, n_1, n_2, \ldots, n_{d-1}\rangle_s, \tag{15}$$

where in the above notation we mean that each phase-shift $\phi_j$ can take $N+1$ discrete values $\phi_j(k_j) = 2k_j\pi/(N+1)$, with $k_j = 0, \ldots, N$. This can be easily checked by using the identities $\sum_{j=0}^{N} e^{2\pi ijk/(N+1)} = (N+1)\delta_{k,0}$ for $-N \le k \le N$. Notice that the identity is guaranteed to hold if the number of discrete phases is at least $N+1$ for each phase-shift to be estimated.

In order to compare the efficiency of multiple phase estimation with the universal case, we consider as a figure of merit the fidelity of the reconstructed state, which corresponds to a cost function of the form $1-F$, where $F$ is the fidelity of the estimated state $|\psi(\{\bar{\phi}_j\})\rangle$ with respect to the true state $|\psi(\{\phi_j\})\rangle$. This cost belongs to the class (8), and therefore the corresponding optimal POVM is (12). By the covariance of the procedure we can write the fidelity as

$$F(\{\phi_j\}) = |\langle\psi(\{\phi_j=0\})|\psi(\{\phi_j\})\rangle|^2 = \frac{1}{d^2}\left[d + 2\sum_{j=1}^{d-1}\cos\phi_j + 2\sum_{j>k}\cos(\phi_j-\phi_k)\right]. \tag{16}$$

By exploiting (11), the resulting maximal fidelity takes the explicit form

$$F_{\mathrm{pc}}^{\mathrm{opt}}(N) = \frac{1}{d} + \frac{d-1}{d^{N+1}} \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-n_1-1}\cdots\sum_{n_{d-1}=0}^{N-n_1-n_2-\cdots-1} \frac{N!}{(N-n_1-n_2-\cdots n_{d-1})!n_1!n_2!\cdots n_{d-1}!} \times\sqrt{\frac{N-n_1-n_2-\cdots-n_{d-1}}{n_1+1}}. \tag{17}$$

Notice that, as expected from the fact that in this case some a priori knowledge about the input state is available, the above fidelity is higher than the universal one for any value of $N$ and $d$.

## 4 Estimation of a Single Qudit

In this section we study in more detail the specific case of the state estimation for a single qudit and comment on the corresponding forms of the optimal measurements. In the universal case the optimal estimation fidelity (3) takes the simple form

$$F_{\mathrm{u}}^{\mathrm{opt}}(1) = \frac{2}{d+1}. \tag{18}$$

It can be straightforwardly proved that the above average fidelity can be achieved by a von Neumann measurement in any basis for the qudit and therefore it corresponds to an easily implementable procedure.

The situation is different in the case of multiple phase estimation. The optimal fidelity for a single equatorial qudit takes the explicit form

$$F_{\mathrm{pc}}^{\mathrm{opt}}(1) = \frac{2d-1}{d^2}. \tag{19}$$

As shown in Sect. 3, the above optimal fidelity can be achieved by a finite number of projectors, which take the form (15). In the particular case of a single qudit the number of such projectors is $2^{d-1}$. We can then see that in the case of qubits the optimal phase estimation procedure can be realised by a von Neumann measurement in any basis of equatorial qubit states, such as for example $\{(|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle - |1\rangle)/\sqrt{2}\}$. However, in dimension higher than two the optimal multi-phase estimation cannot be achieved by an orthogonal set of projectors. For example, in the case of qutrits the measurement with minimum number of projectors corresponds to

$$\hat{P} = \frac{1}{4}[|e(0,0)\rangle\langle e(0,0)| + |e(\pi,0)\rangle\langle e(\pi,0)| + |e(0,\pi)\rangle\langle e(0,\pi)| + |e(\pi,\pi)\rangle\langle e(\pi,\pi)|], \tag{20}$$

with

$$\begin{aligned} |e(0,0)\rangle &= |0\rangle + |1\rangle + |2\rangle, \qquad & |e(\pi,0)\rangle &= |0\rangle - |1\rangle + |2\rangle, \\ |e(0,\pi)\rangle &= |0\rangle + |1\rangle - |2\rangle, \qquad & |e(\pi,\pi)\rangle &= |0\rangle - |1\rangle - |2\rangle. \end{aligned} \tag{21}$$

The above projectors correspond to the estimated phases $\{\phi_1 = \phi_2 = 0\}$, $\{\phi_1 = \pi, \phi_2 = 0\}$, $\{\phi_1 = 0, \phi_2 = \pi\}$ and $\{\phi_1 = \pi, \phi_2 = \pi\}$ respectively.

For the sake of completeness in the description of the case of a single qudit, we discuss and compare the performance of universal and multi-phase covariant estimation with a simpler case where we have even more information about the form of the input state and a single parameter needs to be estimated. We consider the explicit case of an equatorial qudit with only one independent phase-shift, namely of the form

$$|\psi(\{\phi_j = j\phi\})\rangle = \frac{1}{\sqrt{d}}(|0\rangle + e^{i\phi}|1\rangle + e^{2i\phi}|2\rangle + \cdots + e^{i(d-1)\phi}|d-1\rangle), \tag{22}$$

where $\phi \in [0, 2\pi]$. This example can be easily phrased in the framework of a single phase estimation [12, 15, 17] and can be reduced to the optimisation problem of quantum networks discussed in [19]. In this case the optimal POVM for cost functions of the Holevo class can be written in the form

$$d\mu(\phi) = \frac{d\phi}{2\pi}|e(\phi)\rangle\langle e(\phi)|, \tag{23}$$

where the vectors $|e(\phi)\rangle$ are simply given by

$$|e(\phi)\rangle = \sum_{n=0}^{d-1} e^{in\phi}|n\rangle. \tag{24}$$

The corresponding discretised form, which achieves the same efficiency as the above continuous one, can be achieved by a set of $d$ orthogonal projectors $\{|e(2\pi k/d)\rangle, k = 0, \ldots, d-1\}$ and therefore corresponds to a von Neumann measurement. The optimal fidelity takes the explicit form

$$F_1 = \frac{2}{3} + \frac{1}{3d^2}. \tag{25}$$

As expected, for $d > 2$ this value is higher than the cases of universal and multiple phase estimation. We want to point out that the above expression corresponds to the optimal reconstruction fidelity for state estimation of a single qudit in state (22). If we consider the cost function $C(\phi) = \sin^2 \phi/2$, which corresponds to a possible figure of merit if we want to estimate the phase-shift $\phi$ induced by a transformation of the form $\exp(i \sum_{j=1}^{d-1} j \hat{H}_j \phi)$ on the equatorial state of a single qudit, the optimal POVM is still given by (24) and the minimum cost is simply given by $\bar{C} = 1/2d$, namely it vanishes as the inverse of the dimension for increasing dimension, as expected.

## 5 Conclusions

We will now summarise the main points of this work. The study of the precision achievable in the estimation of finite dimensional quantum states and of the possibility to achieve the optimal performance by easily implementable measurement procedures represent a topic of considerable interest, in view also of the big experimental effort which is currently in progress to generate, manipulate and detect quantum systems in dimension higher than two [8, 9]. In this work we have first reviewed some results related to the limits on the optimal estimation procedures for finite dimensional systems in the universal and multi-phase covariant cases. We stress that the present paper does not provide an exhaustive and complete review on state estimation on finite dimensional systems. We have reported the optimal reconstruction fidelity in both cases, even if the procedure outlined for the multi-phase covariant case holds for estimation strategies based on a wide class of cost functions, of which the reconstruction fidelity is only a particular case.

We have then showed that in the multi-phase covariant case a realisation of the optimal estimation procedure can be achieved by a finite set of projectors. We have finally discussed the explicit case of a single qudit and pointed out that the universal optimal estimation can be achieved by a simple von Neumann measurement performed in any single qudit basis, while in the multi-phase covariant case a von Neumann measurement corresponds to an optimal strategy only in the particular case of a single qubit. We have finally reported the optimal fidelity reconstruction for a single phase estimation on a qudit.

**Acknowledgements** This work has been supported in part by EC under the project SECOQC (Contract No. IST-2003-506813) and by the Italian MIUR under PRIN 2005.

## References

1. Chefles, A.: Contemp. Phys. **41**, 401 (2000)
2. Cleve, R., Ekert, A., Macchiavello, C., Mosca, M.: Proc. R. Soc. Lond. A **454**, 339 (1998)
3. Bollinger, J.J., Itano, W.M., Wineland, D.J., Heinzen, D.J.: Phys. Rev. A **54**, R4649 (1996)
4. Bechmann-Pasquinucci, H., Peres, A.: Phys. Rev. Lett. **85**, 3313 (2000)
5. Bruß, D., Macchiavello, C.: Phys. Rev. Lett. **88**, 127901 (2002)

6. Caruso, F., Bechmann-Pasquinucci, H., Macchiavello, C.: Phys. Rev. A **72**, 032340 (2005)
7. Fitzi, M., Gisin, N., Maurer, U.: Phys. Rev. Lett. **87**, 217901 (2002)
8. Molina-Terriza, G., et al.: Phys. Rev. Lett. **92**, 167903 (2004)
9. Moreva, E.V., et al.: Phys. Rev. Lett. **72**, 023602 (2006)
10. Buscemi, F., D'Ariano, G.M., Macchiavello, C.: J. Opt. Soc. Am. B **24**, 363 (2007)
11. Bae, J., Acin, A.: Phys. Rev. Lett. **97**, 030402 (2006)
12. Helstrom, C.W.: Quantum Detection and Estimation Theory. Academic Press, New York (1976)
13. Macchiavello, C.: Phys. Rev. A **67**, 062302 (2003)
14. D'Ariano, G.M., Macchiavello, C., Sacchi, M.F.: Phys. Lett. A **248**, 103 (1998)
15. Holevo, A.S.: Probabilistic and Statistical Aspects of Quantum Theory. North-Holland, Amsterdam (1982)
16. Werner, R.F.: Phys. Rev. A **58**, 1827 (1998)
17. Derka, R., Bužek, V., Ekert, A.: Phys. Rev. Lett. **80**, 1571 (1998)
18. Bruss, D., Macchiavello, C.: Phys. Lett. A **253**, 249 (1999)
19. van Dam, W. et al.: Phys. Rev. Lett. **98**, 090501 (2007)